

MULTIVARIATE FORCE OF MORTALITY

ARNI S.R. SRINIVASA RAO

Bayesian and Interdisciplinary Research Unit,
Indian Statistical Institute,
203 B.T. Road, Kolkata 700108.
Email: arni@isical.ac.in, Tel: +91-33-25753511

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ABSTRACT. We introduce bi-variate and multivariate force of mortality functions. The pattern of mortality in a population is one of the strong influencing factors in determining the life expectancies at various ages in the population. The reasons behind declining forces of mortality could be studied using the proposed functions.

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1. INTRODUCTION

One variable force of mortality, $\mu(x)$, with respect to age, x of an individual is one of the central topics of study in the actuarial mathematics[SmithAMM1948]. It is often termed as instantaneous rate of death at an age x . Suppose, forces of mortality is measured on two variables (x, y) , (one being age, other can be some influencing factor on mortality), then if we plot the force of mortality $\mu(x, y)$ on the xy -plane, then for an arbitrary point (x_0, y_0) , we can write,

$$(1.1) \quad \mu(x_0, y_0) = \frac{1}{\rho(\Omega)} \int \int_{\Omega} \mu(x, y) d\rho$$

where Ω is a region such that $(x_0, y_0) \in \Omega$ and $\rho(\Omega)$ is area of the region. Since μ is continuous in the univariate case, if we assume the same holds for $\mu(x, y)$ in Ω , it has an upper bound say, μ_1 and a lower bound say, μ_0 in the region such that

$$\mu_0 \leq \frac{1}{\rho(\Omega)} \int \int_D \mu(x, y) d\rho \leq \mu_1.$$

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Multivariate force of mortality functions can help in better understanding of future longevity and causes of decline in mortality rates. There are studies which consider mortality decline or longevity projections of humans with respect to age only (for example, see [TuljaLiBoeNature2000, WilmothScience1998]). Such studies can be handled using univariate analysis of standard force of mortality functions. However such studies can be extended to incorporate several variables that can explain decline in mortality rates or causes of increase in longevity using multivariate forces of mortality functions. Further analytical properties such as rate of change in forces of mortalities of bi-variate force of mortality functions are found in the next sections. We will begin with two examples and we illustrate their numerical properties.

Example 1. Suppose $l(x, y) = 1 - x^a y^{b/\sqrt{K}}/K$ for $a, b, K \in \mathbb{Z}^+$. See the definitions of $\mu_x(x, y)$ and $\mu_y(x, y)$ in eq. (3.6) and eq. (3.7) in the section 3. We have,

$$\begin{aligned}\mu_x(x, y) &= \frac{ax^{a-1}y^{b/\sqrt{K}}}{K - x^a y^{b/\sqrt{K}}} \\ \mu_y(x, y) &= \frac{bx^a y^{(b/\sqrt{K})-1}}{\sqrt{K} (K - x^a y^{b/\sqrt{K}})}\end{aligned}$$

Example 2. Suppose $l(x, y) = a\sqrt{y}b^{x^3}$ for $10 \leq x \leq 80$, $1 \leq y \leq 5$, $a = 5$, $b = 7$. $\mu_x(x, y) = -3x^2 \log(b)$ and $\mu_y(x, y) = -\log(a)/2\sqrt{y}$.

Force of mortality functions also associated with continuous life table functions[SmithAMM1948]. Life table is a mathematical model describing how individuals born at same time (for example, a cohort of new born babies) survive over the years at various ages until the last individual dies. Life table is constructed based on present or past mortality pattern (i.e. mortality rates at each age, (say x) per fixed number of individuals in the same age x per year or for the year $(0, t)$) in the population and assumed that this pattern will remain the same until the individual at last age dies. This table can be used to construct synthetic population at each age x at time t or for time interval $(0, t)$, it do not have mechanism to take care of future changes in the mortality pattern after t . Suppose $l(x)$ denote the number of individuals at age x out of $l(0)$ newly born individuals in a life table with continuous partial derivatives up to order $(k+1)$. Then the number of individuals at ages, $x + \Delta x$ and $x - \Delta x$ are denoted by $l(x + \Delta x)$ and $l(x - \Delta x)$ can be obtained from Taylor series expansion evaluating at x_0 as

$$\begin{aligned}l(x_0 + \Delta x) &= l(x_0) + \Delta x l'(x_0) + (\Delta x)^2 \frac{l^{(2)}(x_0)}{2!} + (\Delta x)^3 \frac{l^{(3)}(x_0)}{3!} + \dots \\ &+ (\Delta x)^k \frac{l^{(k)}(x_0)}{k!} + \int_{x_0}^x \frac{(\Delta x)^k l^{(k+1)}(t)}{k!} dt\end{aligned}\tag{1.2}$$

$$\begin{aligned}l(x_0 - \Delta x) &= l(x_0) - \Delta x l'(x_0) + (\Delta x)^2 \frac{l^{(2)}(x_0)}{2!} - (\Delta x)^3 \frac{l^{(3)}(x_0)}{3!} + \dots \\ &+ (-1)^k (\Delta x)^k \frac{l^{(k)}(x_0)}{k!} + \int_{x_0}^x \frac{(\Delta x)^k l^{(k+1)}(t)}{k!} dt\end{aligned}\tag{1.3}$$

Here x is the fixed age between 0 and ω , the maximum age of life. The number of survivors at age x with some other relevant factor (for example, marital status, education level, climate, food habits,

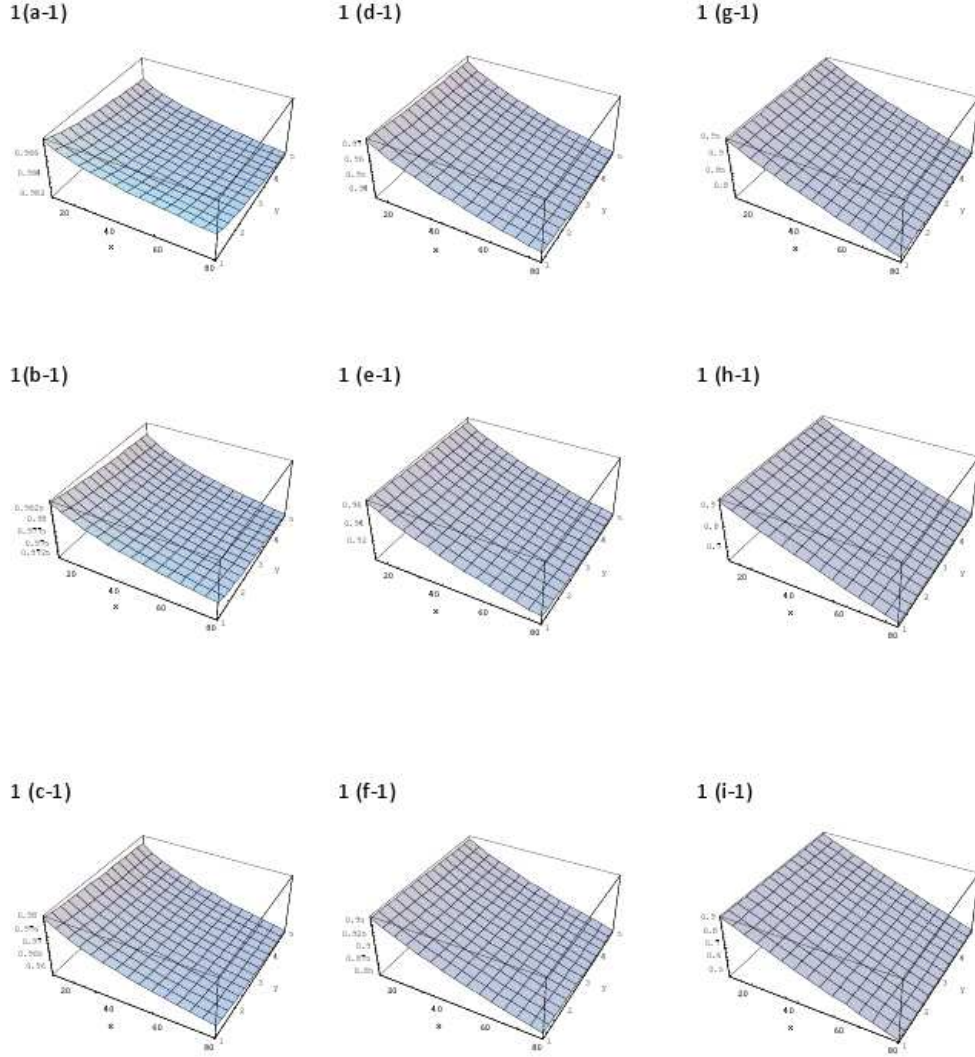


FIGURE 1.1. $l(x, y)$ in Example 1. For various combinations of values of a and b we have drawn $1(a - 1)$ to $1(i - 1)$ by fixing $K = 100$, $10 \leq x \leq 80$ and $1 \leq y \leq 5$. Following are the combinations of a and b for each figure: $1(a - 1)$: $a = 0.1, b = 0.9$, $1(b - 1)$: $a = 0.2, b = 0.8$, $1(c - 1)$: $a = 0.3, b = 0.7$, $1(d - 1)$: $a = 0.4, b = 0.6$, $1(e - 1)$: $a = 0.5, b = 0.5$, $1(f - 1)$: $a = 0.6, b = 0.4$, $1(g - 1)$: $a = 0.7, b = 0.3$, $1(h - 1)$: $a = 0.8, b = 0.2$, $1(i - 1)$: $a = 0.9, b = 0.1$.

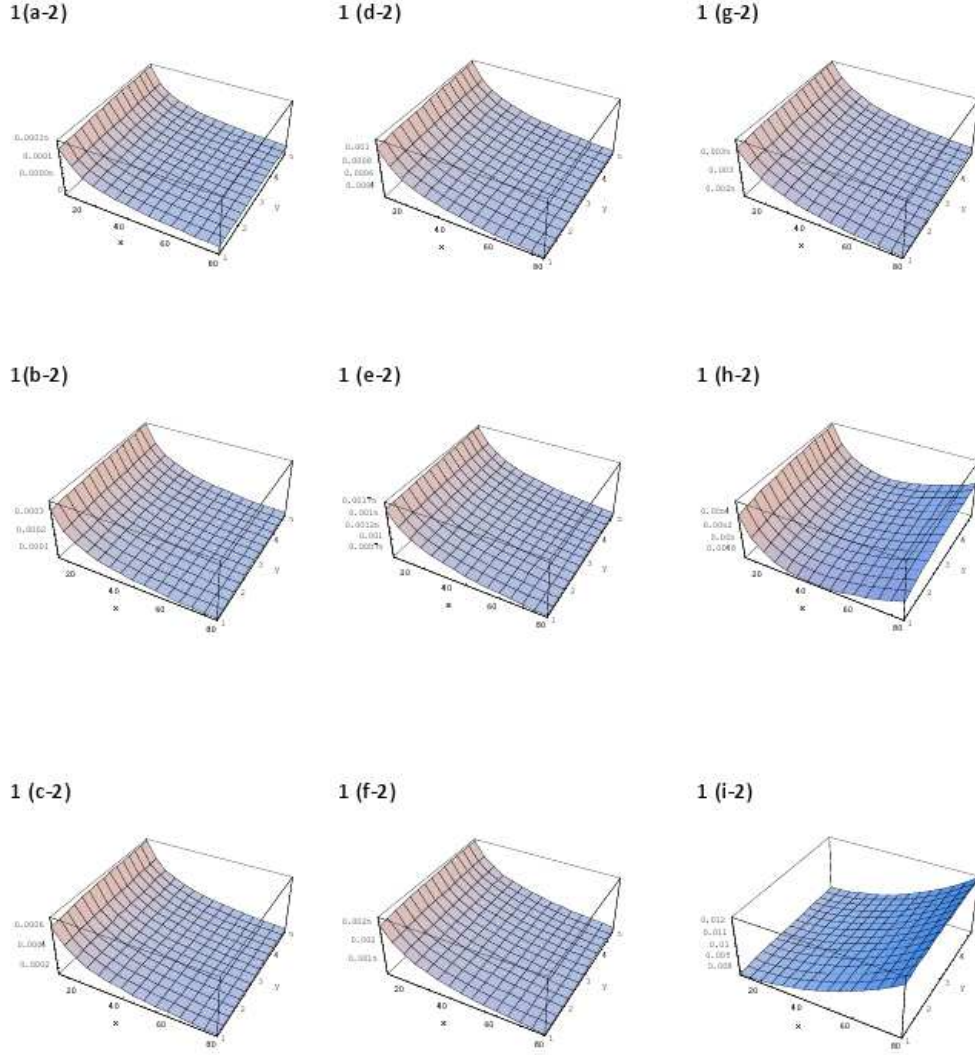


FIGURE 1.2. $\mu_x(x, y)$ in Example 1. For various combinations of values of a and b we have drawn 1(a-1) to 1(i-1) by fixing $K = 100$, $10 \leq x \leq 80$ and $1 \leq y \leq 5$. Following are the combinations of a and b for each figure: 1(a-1) : $a = 0.1, b = 0.9$, 1(b-1) : $a = 0.2, b = 0.8$, 1(c-1) : $a = 0.3, b = 0.7$, 1(d-1) : $a = 0.4, b = 0.6$, 1(e-1) : $a = 0.5, b = 0.5$, 1(f-1) : $a = 0.6, b = 0.4$, 1(g-1) : $a = 0.7, b = 0.3$, 1(h-1) : $a = 0.8, b = 0.2$, 1(i-1) : $a = 0.9, b = 0.1$.

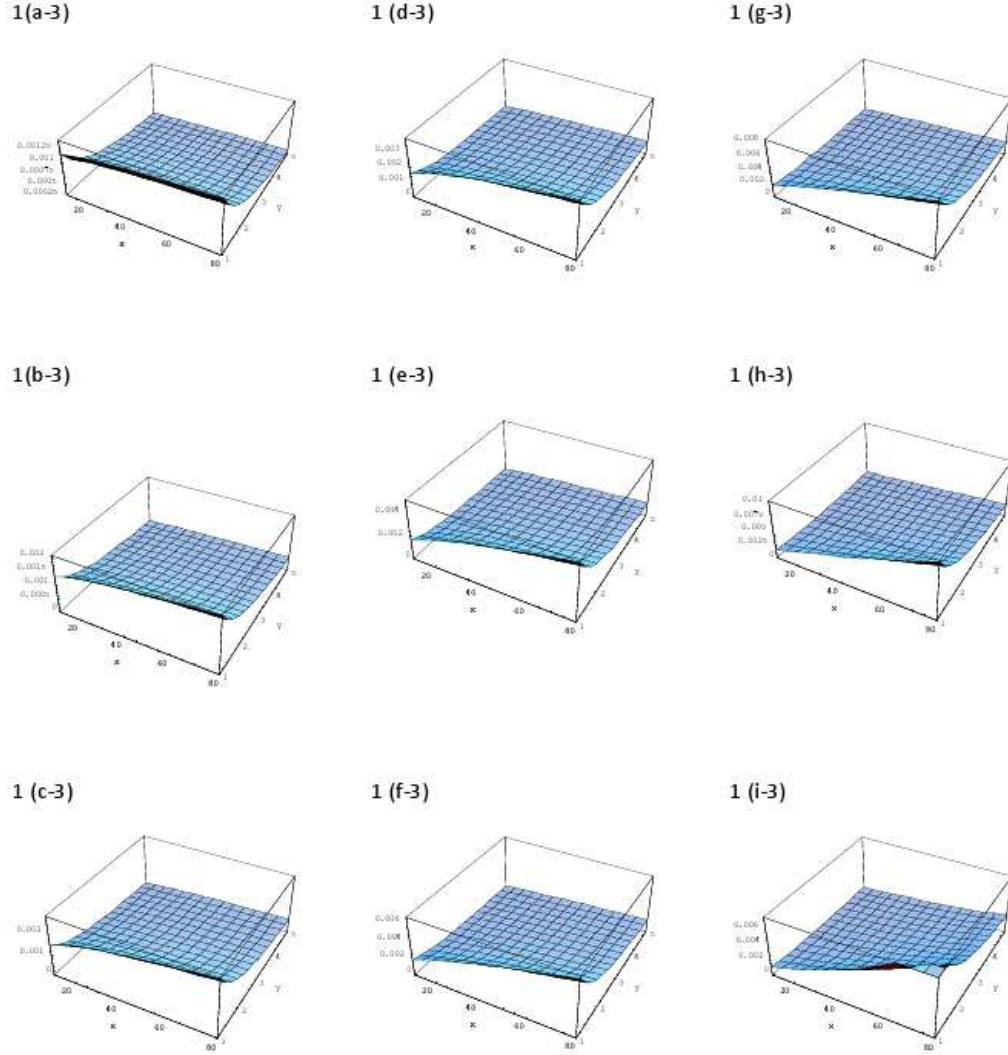


FIGURE 1.3. $\mu_y(x, y)$ in Example 1. For various combinations of values of a and b we have drawn $1(a - 1)$ to $1(i - 1)$ by fixing $K = 100$, $10 \leq x \leq 80$ and $1 \leq y \leq 5$. Following are the combinations of a and b for each figure: $1(a - 1)$: $a = 0.1, b = 0.9$, $1(b - 1)$: $a = 0.2, b = 0.8$, $1(c - 1)$: $a = 0.3, b = 0.7$, $1(d - 1)$: $a = 0.4, b = 0.6$, $1(e - 1)$: $a = 0.5, b = 0.5$, $1(f - 1)$: $a = 0.6, b = 0.4$, $1(g - 1)$: $a = 0.7, b = 0.3$, $1(h - 1)$: $a = 0.8, b = 0.2$, $1(i - 1)$: $a = 0.9, b = 0.1$.

geographic region etc) y evaluated at (x_0, y_0) can be obtained from two variable Taylor expansion. Assuming continuous partial derivatives for $l(x_0 + \Delta x, y_0 + \Delta y)$ up to order 3, we can the expansion of two variable survival functions as follows:

$$\begin{aligned}
 l(x_0 + \Delta x, y_0 + \Delta y) &= l(x_0, y_0) + \Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} + \Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} + \frac{(\Delta x)^2}{2!} \left\{ \frac{\partial^2 l}{\partial x^2}(x_0, y_0) \right\} \\
 &+ \frac{(\Delta y)^2}{2!} \left\{ \frac{\partial^2 l}{\partial y^2}(x_0, y_0) \right\} + \Delta x \Delta y \left\{ \frac{\partial^2 l}{\partial x \partial y}(x_0, y_0) \right\} + \frac{(\Delta x)^3}{3!} \left\{ \frac{\partial^3 l}{\partial x^3}(x_0, y_0) \right\} \\
 &+ \frac{(\Delta y)^3}{3!} \left\{ \frac{\partial^3 l}{\partial y^3}(x_0, y_0) \right\} + \frac{(\Delta x)^2 \Delta y}{2} \left\{ \frac{\partial^3 l}{\partial x^2 \partial y}(x_0, y_0) \right\} \\
 &+ \frac{\Delta x (\Delta y)^2}{2} \left\{ \frac{\partial^3 l}{\partial x \partial y^2}(x_0, y_0) \right\} + \dots + \dots
 \end{aligned}
 \tag{1.4}$$

$$\begin{aligned}
 l(x_0 - \Delta x, y_0 - \Delta y) &= l(x_0, y_0) - \Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} - \Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} + \frac{(\Delta x)^2}{2!} \left\{ \frac{\partial^2 l}{\partial x^2}(x_0, y_0) \right\} \\
 &+ \frac{(\Delta y)^2}{2!} \left\{ \frac{\partial^2 l}{\partial y^2}(x_0, y_0) \right\} + \Delta x \Delta y \left\{ \frac{\partial^2 l}{\partial x \partial y}(x_0, y_0) \right\} - \frac{(\Delta x)^3}{3!} \left\{ \frac{\partial^3 l}{\partial x^3}(x_0, y_0) \right\} \\
 &- \frac{(\Delta y)^3}{3!} \left\{ \frac{\partial^3 l}{\partial y^3}(x_0, y_0) \right\} - \frac{(\Delta x)^2 \Delta y}{2} \left\{ \frac{\partial^3 l}{\partial x^2 \partial y}(x_0, y_0) \right\} \\
 &- \frac{\Delta x (\Delta y)^2}{2} \left\{ \frac{\partial^3 l}{\partial x \partial y^2}(x_0, y_0) \right\} + \dots + \dots
 \end{aligned}
 \tag{1.5}$$

2. ANALYSIS OF FIRST ORDER EQUATIONS

We try to analyze univariate force of mortality functions by considering higher order derivatives are continuous. Assuming $x \rightarrow x_0$, in eq. (1.2) and eq. (1.3), we will have $\int_{x_0}^x ((\Delta x)^k l^{(k+1)}(t)/k!) dt \rightarrow 0$. Suppose $f^{(n)}(x_0 + \Delta x)$ and $f^{(n)}(x_0 - \Delta x)$ denote Taylor expansion equations when ignoring the $(n+1)^{th}$ derivatives and beyond for $n = 2, 3, \dots$ in eq. (1.2) and eq. (1.3), then by sequentially ignoring the terms beginning from the term $\frac{(\Delta x)^n}{n!} l^{(n)}(x_0)$ in eq. (1.2) and eq. (1.3) for $n = 2, 3, \dots$, we will obtain following equations:

$$f^{(1)}(x_0 + \Delta x) - f^{(1)}(x_0 - \Delta x) = 2\Delta x l'(x_0)
 \tag{2.1}$$

$$f^{(3)}(x_0 + \Delta x) - f^{(3)}(x_0 - \Delta x) = 2\Delta x l'(x_0) + \frac{2(\Delta x)^3}{3!} l^{(3)}(x_0)
 \tag{2.2}$$

$$\begin{aligned}
f^{(5)}(x_0 + \Delta x) - f^{(5)}(x_0 - \Delta x) &= 2\Delta x l'(x_0) + \frac{2(\Delta x)^3}{3!} l^{(3)}(x_0) + \frac{2(\Delta x)^5}{5!} l^{(5)}(x_0) \\
&\vdots \\
f^{(2k-1)}(x_0 + \Delta x) - f^{(2k-1)}(x_0 - \Delta x) &= 2\Delta x l'(x_0) + \frac{2(\Delta x)^3}{3!} l^{(3)}(x_0) + \dots + \frac{2(\Delta x)^{(2k-1)}}{(2k-1)!} l^{(2k-1)}(x_0) \\
&\vdots
\end{aligned}
\tag{2.3}$$

Therefore,

$$\begin{aligned}
&\sum_{j=1}^k [\{f^{(2j+1)}(x_0 + \Delta x) - f^{(2j+1)}(x_0 - \Delta x)\} - \{f^{(2j-1)}(x_0 + \Delta x) - f^{(2j-1)}(x_0 - \Delta x)\}] = \\
&\frac{2(\Delta x)^3}{3!} l^{(3)}(x_0) + \frac{2(\Delta x)^5}{5!} l^{(5)}(x_0) + \dots + \frac{2(\Delta x)^{(2k+1)}}{(2k+1)!} l^{(2k+1)}(x_0)
\end{aligned}
\tag{2.4}$$

Hence we will obtain,

$$\begin{aligned}
&\sum_{j=1}^k [\{f^{(2j+1)}(x_0 + \Delta x) - f^{(2j+1)}(x_0 - \Delta x)\} - \{f^{(2j-1)}(x_0 + \Delta x) - f^{(2j-1)}(x_0 - \Delta x)\}] = \\
&\{f^{(2k+1)}(x_0 + \Delta x) - f^{(2k+1)}(x_0 - \Delta x)\} - \{f^{(1)}(x_0 + \Delta x) - f^{(1)}(x_0 - \Delta x)\}
\end{aligned}
\tag{2.5}$$

Using the relation $l^{(1)}(x_0) = \frac{1}{2\Delta x} [f^{(1)}(x_0 + \Delta x) - f^{(1)}(x_0 - \Delta x)]$, we write,

$$\begin{aligned}
\left. \frac{dl(x)}{dx} \right|_{x=x_0} &= \frac{1}{2\Delta x} \left[\{f^{(2k+1)}(x_0 + \Delta x) - f^{(2k+1)}(x_0 - \Delta x)\} - \right. \\
&\left. \sum_{j=1}^k [\{f^{(2j+1)}(x_0 + \Delta x) - f^{(2j+1)}(x_0 - \Delta x)\} - \{f^{(2j-1)}(x_0 + \Delta x) - f^{(2j-1)}(x_0 - \Delta x)\}] \right]
\end{aligned}
\tag{2.6}$$

Dividing the eq. (2.1) by the term $2\Delta x$ on both the sides and integrating it from age x to $x + m$, we obtain,

$$\begin{aligned}
\int_x^{x+m} \frac{\{f^{(1)}(y_0 + \Delta y) - f^{(1)}(y_0 - \Delta y)\}}{2\Delta y} dy &= \int_x^{x+m} \left. \frac{d}{dy} l(y) \right|_{y=y_0} dy \\
&= - \int_x^{x+m} \left. \frac{1}{l(y)} \frac{d}{dy} l(y) \right|_{y=y_0} l(y)|_{y=y_0} dy \\
&= - \int_x^{x+m} \mu(y)|_{y=y_0} l(y)|_{y=y_0} dy
\end{aligned}
\tag{2.7}$$

where, $\mu(x)$, the force of mortality function, which is defined as $(-1/l(x)) (d/dx)l(x)$.

Dividing the eq. (2.2) by the term $2\Delta x$ on both the sides and integrating it from age x to $x + m$, we obtain,

$$\int_x^{x+m} \left[\frac{\{f^{(3)}(y_0 + \Delta y) - f^{(3)}(y_0 - \Delta y)\}}{2\Delta y} - \frac{(\Delta y)^2}{3!} l^{(3)}(y_0) \right] dy = - \int_x^{x+m} \mu(y)|_{y=y_0} l(y)|_{y=y_0} dy
\tag{2.8}$$

Now, multiplying $l(x)|_{x=x_0}$ and $(-1/l(x))|_{x=x_0}$ to the eq. (2.6), and integrating between ages x and $x+m$, we will obtain,

$$\begin{aligned}
 - \int_x^{x+m} -\frac{1}{l(y)} \frac{d}{dy} l(y) \Big|_{y=y_0} l(y)|_{y=y_0} dy &= \int_x^{x+m} -\frac{1}{2\Delta y} \left[\{f^{(2k+1)}(y_0 + \Delta y) - f^{(2k+1)}(y_0 - \Delta y)\} \right. \\
 &\quad - \sum_{j=1}^k [\{f^{(2j+1)}(y_0 + \Delta y) - f^{(2j+1)}(y_0 - \Delta y)\} \\
 &\quad \left. - \{f^{(2j-1)}(y_0 + \Delta y) - f^{(2j-1)}(y_0 - \Delta y)\} \right] \\
 &= - \int_x^{x+m} \frac{d}{dy} l(y) \Big|_{y=y_0} dy \\
 (2.9) \qquad &= ml'(y_0)
 \end{aligned}$$

If there are any deaths during the age $(x_0 + \Delta x)$ to $(x_0 - \Delta x)$, then $f^{(1)}(x_0 + \Delta x) < f^{(1)}(x_0 - \Delta x)$. In the absence of deaths, we have $f^{(1)}(x_0 + \Delta x) = f^{(1)}(x_0 - \Delta x)$ and $\frac{dl(x)}{dx} \Big|_{x=x_0} < 0$. This argument is true for difference of other higher order expressions.

3. ANALYSIS OF SECOND ORDER EQUATIONS

Suppose $g^{(n)}(x_0 + \Delta x, y_0 + \Delta y)$ and $g^{(n)}(x_0 - \Delta x, y_0 - \Delta y)$ denote equations when ignoring the terms from $(n+1)^{th}$ partial derivatives and beyond for $n = 2, 3, \dots$ in eq. (1.4) and eq. (1.5), then by sequentially ignoring the terms beginning from the n^{th} order partial derivative terms in eq. (1.4) and eq. (1.5) for $n = 2, 3, \dots$, we will obtain following equations:

$$\begin{aligned}
 g^{(1)}(x_0 + \Delta x, y_0 + \Delta y) - g^{(1)}(x_0 - \Delta x, y_0 - \Delta y) &= 2\Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} + 2\Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} \\
 g^{(3)}(x_0 + \Delta x, y_0 + \Delta y) - g^{(3)}(x_0 - \Delta x, y_0 - \Delta y) &= 2\Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} + 2\Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} \\
 &\quad + \frac{(\Delta x)^3}{3!} \left\{ \frac{\partial^3 l}{\partial x^3}(x_0, y_0) \right\} + \frac{(\Delta y)^3}{3!} \left\{ \frac{\partial^3 l}{\partial y^3}(x_0, y_0) \right\} \\
 &\quad + \frac{(\Delta x)^2 \Delta y}{2} \left\{ \frac{\partial^3 l}{\partial x^2 \partial y}(x_0, y_0) \right\} \\
 &\quad + \frac{\Delta x (\Delta y)^2}{2} \left\{ \frac{\partial^3 l}{\partial x \partial y^2}(x_0, y_0) \right\} \\
 &\quad \vdots \quad \vdots
 \end{aligned}$$

$$\begin{aligned}
g^{(2k-1)}(x_0 + \Delta x, y_0 + \Delta y) - g^{(2k-1)}(x_0 - \Delta x, y_0 - \Delta y) &= 2\Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} + 2\Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} + \dots \\
&\quad + \frac{2(\Delta x)^{2k-j-1}(\Delta y)^j}{(2k-1)!} \left[\sum_{j=0}^{2k-1} \binom{2k-1}{j} \times \right. \\
&\quad \left. \left\{ \frac{\partial^{(2k-1)} l}{\partial x^{(2k-j-1)} \partial y^j}(x_0, y_0) \right\} \right] \\
&\quad \vdots \quad \quad \quad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{j=1}^k \left[\{g^{(2k+1)}(x_0 + \Delta x, y_0 + \Delta y) - g^{(2k+1)}(x_0 - \Delta x, y_0 - \Delta y)\} \right. \\
&\quad \left. - \{g^{(2k-1)}(x_0 + \Delta x, y_0 + \Delta y) - g^{(2k-1)}(x_0 - \Delta x, y_0 - \Delta y)\} \right] = \frac{(\Delta x)^3}{3!} \left\{ \frac{\partial^3 l}{\partial x^3}(x_0, y_0) \right\} \\
&\quad + \frac{(\Delta y)^3}{3!} \left\{ \frac{\partial^3 l}{\partial y^3}(x_0, y_0) \right\} + \frac{(\Delta x)^2 \Delta y}{2} \left\{ \frac{\partial^3 l}{\partial x^2 \partial y}(x_0, y_0) \right\} + \frac{\Delta x (\Delta y)^2}{2} \left\{ \frac{\partial^3 l}{\partial x \partial y^2}(x_0, y_0) \right\} + \\
(3.1) \quad &\dots + \frac{2(\Delta x)^{2k-j+1}(\Delta y)^j}{(2k+1)!} \left[\sum_{j=0}^{2k+1} \binom{2k+1}{j} \left\{ \frac{\partial^{(2k+1)} l}{\partial x^{(2k-j+1)} \partial y^j}(x_0, y_0) \right\} \right]
\end{aligned}$$

Force of mortality for the two variables (x, y) is evaluated at the point (x_0, y_0) using partial derivatives as follows:

$$(3.2) \quad \frac{\partial \mu(x, y)}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{\mu(x_0 + \Delta x, y_0) - \mu(x_0, y_0)}{\Delta x}$$

$$(3.3) \quad \frac{\partial \mu(x, y)}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{\mu(x_0, y_0 + \Delta y) - \mu(x_0, y_0)}{\Delta y}$$

where, we define,

$$(3.4) \quad \mu(x_0 + \Delta x, y_0) = -\frac{1}{l(x_0 + \Delta x, y_0)} \frac{\partial l(x_0 + \Delta x, y_0)}{\partial x}$$

$$(3.5) \quad \mu(x_0, y_0 + \Delta y) = -\frac{1}{l(x_0, y_0 + \Delta y)} \frac{\partial l(x_0, y_0 + \Delta y)}{\partial y}$$

$$(3.6) \quad \mu_x(x_0, y_0) = -\frac{1}{l(x_0, y_0)} \frac{\partial l(x_0, y_0)}{\partial x} \text{ for eq. (3.2)}$$

$$(3.7) \quad \mu_y(x_0, y_0) = -\frac{1}{l(x_0, y_0)} \frac{\partial l(x_0, y_0)}{\partial y} \text{ for eq. (3.3)}$$

Using eq. (1.4) and as $\Delta y \rightarrow 0$, we obtain,

$$l(x_0 + \Delta x, y_0) = l(x_0, y_0) + \Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} + \frac{(\Delta x)^2}{2!} \left\{ \frac{\partial^2 l}{\partial x^2}(x_0, y_0) \right\} + \frac{(\Delta x)^3}{3!} \left\{ \frac{\partial^3 l}{\partial x^3}(x_0, y_0) \right\} \quad (3.8)$$

Using eq. (1.5) and as $\Delta x \rightarrow 0$, we obtain,

$$l(x_0, y_0 + \Delta y) = l(x_0, y_0) + \Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} + \frac{(\Delta y)^2}{2!} \left\{ \frac{\partial^2 l}{\partial y^2}(x_0, y_0) \right\} + \frac{(\Delta y)^3}{3!} \left\{ \frac{\partial^3 l}{\partial y^3}(x_0, y_0) \right\} \quad (3.9)$$

Therefore,

$$\frac{\partial l}{\partial x}(x_0 + \Delta x, y_0) = \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} + \Delta x \left\{ \frac{\partial^2 l}{\partial x^2}(x_0, y_0) \right\} + \frac{(\Delta x)^2}{2!} \left\{ \frac{\partial^3 l}{\partial x^3}(x_0, y_0) \right\} + \frac{(\Delta x)^3}{3!} \left\{ \frac{\partial^4 l}{\partial x^4}(x_0, y_0) \right\} \quad (3.10)$$

$$\frac{\partial l}{\partial y}(x_0, y_0 + \Delta y) = \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} + \Delta y \left\{ \frac{\partial^2 l}{\partial y^2}(x_0, y_0) \right\} + \frac{(\Delta y)^2}{2!} \left\{ \frac{\partial^3 l}{\partial y^3}(x_0, y_0) \right\} + \frac{(\Delta y)^3}{3!} \left\{ \frac{\partial^4 l}{\partial y^4}(x_0, y_0) \right\} \quad (3.11)$$

Let us now derive the equation of the type eq. (3.1) with the conditions $\Delta y \rightarrow 0$ and $\Delta x \rightarrow 0$ and extending these equations up to the general term. Suppose, $\Delta y \rightarrow 0$ in the equations (1.4) and (1.5), then

$$l(x_0 + \Delta x, y_0) = l(x_0, y_0) + \Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} + \frac{(\Delta x)^2}{2!} \left\{ \frac{\partial^2 l}{\partial x^2}(x_0, y_0) \right\} + \dots + \frac{(\Delta x)^n}{n!} \left\{ \frac{\partial^n l}{\partial x^n}(x_0, y_0) \right\} + \dots \quad (3.12)$$

$$l(x_0 - \Delta x, y_0) = l(x_0, y_0) - \Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} + \frac{(\Delta x)^2}{2!} \left\{ \frac{\partial^2 l}{\partial x^2}(x_0, y_0) \right\} - \dots + (-1)^n \frac{(\Delta x)^n}{n!} \left\{ \frac{\partial^n l}{\partial x^n}(x_0, y_0) \right\} + \dots \quad (3.13)$$

and $\Delta x \rightarrow 0$ in the equations (1.4) and (1.5), then

$$l(x_0, y_0 + \Delta y) = l(x_0, y_0) + \Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} + \frac{(\Delta y)^2}{2!} \left\{ \frac{\partial^2 l}{\partial y^2}(x_0, y_0) \right\} + \dots + \frac{(\Delta y)^n}{n!} \left\{ \frac{\partial^n l}{\partial y^n}(x_0, y_0) \right\} + \dots \quad (3.14)$$

$$l(x_0, y_0 - \Delta y) = l(x_0, y_0) - \Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} + \frac{(\Delta y)^2}{2!} \left\{ \frac{\partial^2 l}{\partial y^2}(x_0, y_0) \right\} - \dots + (-1)^n \frac{(\Delta y)^n}{n!} \left\{ \frac{\partial^n l}{\partial y^n}(x_0, y_0) \right\} + \dots \quad (3.15)$$

Sequentially, ignoring the n^{th} order terms from eq. (3.12) and eq. (3.13), and denoting these new equations as $g_1^{(n)}(x_0 + \Delta x, y_0)$ and $g_1^{(n)}(x_0 - \Delta x, y_0)$ for $n = 2, 3, \dots$, we obtain below set of equations.

$$\begin{aligned}
 (3.16) \quad g_1^{(1)}(x_0 + \Delta x, y_0) - g_1^{(1)}(x_0 - \Delta x, y_0) &= 2\Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} \\
 g_1^{(3)}(x_0 + \Delta x, y_0) - g_1^{(3)}(x_0 - \Delta x, y_0) &= 2\Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} + \frac{2(\Delta x)^3}{3!} \left\{ \frac{\partial^3 l}{\partial x^3}(x_0, y_0) \right\} \\
 &\vdots \\
 g_1^{(2k-1)}(x_0 + \Delta x, y_0) - g_1^{(2k-1)}(x_0 - \Delta x, y_0) &= 2\Delta x \left\{ \frac{\partial l}{\partial x}(x_0, y_0) \right\} + \frac{2(\Delta x)^3}{3!} \left\{ \frac{\partial^3 l}{\partial x^3}(x_0, y_0) \right\} + \\
 &\quad \dots + \frac{2(\Delta x)^{(2k-1)}}{(2k-1)!} \left\{ \frac{\partial^{(2k-1)} l}{\partial x^{(2k-1)}}(x_0, y_0) \right\}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (3.17) \quad \sum_{j=0}^k \left[\left\{ g_1^{(2j+1)}(x_0 + \Delta x, y_0) - g_1^{(2j+1)}(x_0 - \Delta x, y_0) \right\} - \left\{ g_1^{(2j-1)}(x_0 + \Delta x, y_0) - g_1^{(2j-1)}(x_0 - \Delta x, y_0) \right\} \right] &= \\
 \sum_{j=1}^k \frac{2(\Delta x)^{(2j+1)}}{(2j+1)!} \left\{ \frac{\partial^{(2j+1)} l}{\partial x^{(2j+1)}}(x_0, y_0) \right\} &= \\
 \left\{ g_1^{(2k+1)}(x_0 + \Delta x, y_0) - g_1^{(2k+1)}(x_0 - \Delta x, y_0) \right\} - \left\{ g_1^{(1)}(x_0 + \Delta x, y_0) - g_1^{(1)}(x_0 - \Delta x, y_0) \right\}
 \end{aligned}$$

Sequentially, ignoring the n^{th} order terms from eq. (3.14) and eq. (3.15), and denoting these new equations as $g_2^{(n)}(x_0, y_0 + \Delta y)$ and $g_2^{(n)}(x_0, y_0 - \Delta y)$ for $n = 2, 3, \dots$, we obtain below set of equations.

$$\begin{aligned}
 (3.18) \quad g_2^{(1)}(x_0, y_0 + \Delta y) - g_2^{(1)}(x_0, y_0 - \Delta y) &= 2\Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} \\
 g_2^{(3)}(x_0, y_0 + \Delta y) - g_2^{(3)}(x_0, y_0 - \Delta y) &= 2\Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} + \frac{2(\Delta y)^3}{3!} \left\{ \frac{\partial^3 l}{\partial y^3}(x_0, y_0) \right\} \\
 &\vdots \\
 g_2^{(2k-1)}(x_0, y_0 + \Delta y) - g_2^{(2k-1)}(x_0, y_0 - \Delta y) &= 2\Delta y \left\{ \frac{\partial l}{\partial y}(x_0, y_0) \right\} + \frac{2(\Delta y)^3}{3!} \left\{ \frac{\partial^3 l}{\partial y^3}(x_0, y_0) \right\} + \\
 &\quad \dots + \frac{2(\Delta y)^{(2k-1)}}{(2k-1)!} \left\{ \frac{\partial^{(2k-1)} l}{\partial y^{(2k-1)}}(x_0, y_0) \right\}
 \end{aligned}$$

Therefore,

$$\sum_{j=0}^k \left[\left\{ g_2^{(2j+1)}(x_0, y_0 + \Delta y) - g_2^{(2j+1)}(x_0, y_0 - \Delta y) \right\} - \left\{ g_2^{(2j-1)}(x_0, y_0 + \Delta y) - g_2^{(2j-1)}(x_0, y_0 - \Delta y) \right\} \right] =$$

$$\begin{aligned}
(3.19) \quad & \sum_{j=1}^k \frac{2(\Delta y)^{(2j+1)}}{(2j+1)!} \left\{ \frac{\partial^{(2j+1)} l}{\partial y^{(2j+1)}} (x_0, y_0) \right\} \\
&= \left\{ l^{(2k+1)} (x_0, y_0 + \Delta y) - l^{(2k+1)} (x_0, y_0 - \Delta y) \right\} - \left\{ l^{(1)} (x_0, y_0 + \Delta y) - l^{(1)} (x_0, y_0 - \Delta y) \right\}
\end{aligned}$$

Now substituting the eqs. (3.8) and (3.10) in the eq. (3.4), we get

$$(3.20) \quad \mu(x_0 + \Delta x, y_0) = - \frac{1}{\left[l(x_0, y_0) + \sum_{j=1}^{\infty} \frac{(\Delta x)^j}{j!} \left\{ \frac{\partial^j l}{\partial x^j} (x_0, y_0) \right\} \right]} \sum_{j=0}^{\infty} \frac{(\Delta x)^j}{j!} \left\{ \frac{\partial^{j+1} l}{\partial x^{j+1}} (x_0, y_0) \right\}$$

and, substituting the eqs. (3.9) and (3.11) in the eq. (3.5), we get

$$(3.21) \quad \mu(x_0, y_0 + \Delta y) = - \frac{1}{\left[l(x_0, y_0) + \sum_{j=1}^{\infty} \frac{(\Delta y)^j}{j!} \left\{ \frac{\partial^j l}{\partial y^j} (x_0, y_0) \right\} \right]} \sum_{j=0}^{\infty} \frac{(\Delta y)^j}{j!} \left\{ \frac{\partial^{j+1} l}{\partial y^{j+1}} (x_0, y_0) \right\}$$

Using eq. (3.16) and eq. (3.18), we obtain forces of mortalities for two variables as follows:

$$\begin{aligned}
\mu(x_0, y_0) &= - \frac{\left\{ g_1^{(1)}(x_0 + \Delta x, y_0) - g_1^{(1)}(x_0 - \Delta x, y_0) \right\}}{2\Delta x l(x_0, y_0)} \\
&\quad - \frac{\left\{ g_2^{(1)}(x_0, y_0 + \Delta y) - g_2^{(1)}(x_0, y_0 - \Delta y) \right\}}{2\Delta y l(x_0, y_0)}
\end{aligned}$$

Hence the derivative forces of mortalities are as follows:

$$(3.22) \quad \frac{\partial \mu(x, y)}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\left\{ g_1^{(1)}(x_0 + \Delta x, y_0) - g_1^{(1)}(x_0 - \Delta x, y_0) \right\}}{2\Delta x l(x_0, y_0)} - \frac{\sum_{j=0}^{\infty} \frac{(\Delta x)^j}{j!} \left\{ \frac{\partial^{j+1} l}{\partial x^{j+1}} (x_0, y_0) \right\}}{\left[l(x_0, y_0) + \sum_{j=1}^{\infty} \frac{(\Delta x)^j}{j!} \left\{ \frac{\partial^j l}{\partial x^j} (x_0, y_0) \right\} \right]} \right]$$

$$(3.23) \quad \frac{\partial \mu(x, y)}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} \left[\frac{\left\{ g_2^{(1)}(x_0, y_0 + \Delta y) - g_2^{(1)}(x_0, y_0 - \Delta y) \right\}}{2\Delta y l(x_0, y_0)} - \frac{\sum_{j=0}^{\infty} \frac{(\Delta y)^j}{j!} \left\{ \frac{\partial^{j+1} l}{\partial y^{j+1}} (x_0, y_0) \right\}}{\left[l(x_0, y_0) + \sum_{j=1}^{\infty} \frac{(\Delta y)^j}{j!} \left\{ \frac{\partial^j l}{\partial y^j} (x_0, y_0) \right\} \right]} \right]$$

4. ANALYSIS OF THIRD ORDER EQUATIONS

Suppose s be the function number of survivors at age x with two more influencing variables y and z , then the three variable survival function $s(x, y, z)$ evaluated at $\{x_0, y_0, z_0\}$ can be written as

$$\begin{aligned}
s(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) &= s(x_0, y_0, z_0) + \Delta x \left\{ \frac{\partial s}{\partial x}(x_0, y_0, z_0) \right\} + \Delta y \left\{ \frac{\partial s}{\partial y}(x_0, y_0, z_0) \right\} \\
&+ \Delta z \left\{ \frac{\partial s}{\partial z}(x_0, y_0, z_0) \right\} + \frac{\Delta x^2}{2} \left\{ \frac{\partial^2 s}{\partial x^2}(x_0, y_0, z_0) \right\} \\
&+ \frac{\Delta y^2}{2} \left\{ \frac{\partial^2 s}{\partial y^2}(x_0, y_0, z_0) \right\} + \frac{\Delta z^2}{2} \left\{ \frac{\partial^2 s}{\partial z^2}(x_0, y_0, z_0) \right\} \\
&+ \Delta x \Delta y \left\{ \frac{\partial^2 s}{\partial x \partial y}(x_0, y_0, z_0) \right\} + \Delta x \Delta z \left\{ \frac{\partial^2 s}{\partial x \partial z}(x_0, y_0, z_0) \right\} \\
&+ \Delta y \Delta z \left\{ \frac{\partial^2 s}{\partial y \partial z}(x_0, y_0, z_0) \right\} + \\
(4.1) \quad &\dots + \sum_{n=k}^{\infty} \left[\sum_{n_1, n_2, n_3} \frac{1}{n_1! n_2! n_3!} \frac{\partial^k s}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} \Delta x^{n_1} \Delta y^{n_2} \Delta z^{n_3} \right]
\end{aligned}$$

Similarly,

$$\begin{aligned}
s(x_0 - \Delta x, y_0 - \Delta y, z_0 - \Delta z) &= s(x_0, y_0, z_0) - \Delta x \left\{ \frac{\partial s}{\partial x}(x_0, y_0, z_0) \right\} - \Delta y \left\{ \frac{\partial s}{\partial y}(x_0, y_0, z_0) \right\} \\
&- \Delta z \left\{ \frac{\partial s}{\partial z}(x_0, y_0, z_0) \right\} + \frac{\Delta x^2}{2} \left\{ \frac{\partial^2 s}{\partial x^2}(x_0, y_0, z_0) \right\} \\
&+ \frac{\Delta y^2}{2} \left\{ \frac{\partial^2 s}{\partial y^2}(x_0, y_0, z_0) \right\} + \frac{\Delta z^2}{2} \left\{ \frac{\partial^2 s}{\partial z^2}(x_0, y_0, z_0) \right\} \\
&+ \Delta x \Delta y \left\{ \frac{\partial^2 s}{\partial x \partial y}(x_0, y_0, z_0) \right\} + \Delta x \Delta z \left\{ \frac{\partial^2 s}{\partial x \partial z}(x_0, y_0, z_0) \right\} \\
&+ \Delta y \Delta z \left\{ \frac{\partial^2 s}{\partial y \partial z}(x_0, y_0, z_0) \right\} + \\
(4.2) \quad &\dots + (-1)^k \sum_{n=k}^{\infty} \left[\sum_{n_1, n_2, n_3} \frac{1}{n_1! n_2! n_3!} \frac{\partial^k s}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} \Delta x^{n_1} \Delta y^{n_2} \Delta z^{n_3} \right]
\end{aligned}$$

Here $k = n_1 + n_2 + n_3$. Let $h^{(n)}(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$ and $h^{(n)}(x_0 - \Delta x, y_0 - \Delta y, z_0 - \Delta z)$ denote equations after ignoring the terms with derivatives beginning from the $(n+1)^{th}$ order ($n = 1, 2, \dots$) in the eqs. (4.1) and (4.2). We will obtain following equations:

$$\begin{aligned}
\left. \begin{aligned} &h^{(1)}(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) \\ &-h^{(1)}(x_0 - \Delta x, y_0 - \Delta y, z_0 - \Delta z) \end{aligned} \right\} &= 2\Delta x \left\{ \frac{\partial s}{\partial x}(x_0, y_0, z_0) \right\} + 2\Delta y \left\{ \frac{\partial s}{\partial y}(x_0, y_0, z_0) \right\} \\
&+ 2\Delta z \left\{ \frac{\partial s}{\partial z}(x_0, y_0, z_0) \right\} \\
&\vdots \quad \vdots \quad \vdots
\end{aligned}$$

$$\begin{aligned}
\left. \begin{aligned} &h^{(2k-1)}(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) \\ &-h^{(2k-1)}(x_0 - \Delta x, y_0 - \Delta y, z_0 - \Delta z) \end{aligned} \right\} &= 2\Delta x \left\{ \frac{\partial s}{\partial x}(x_0, y_0, z_0) \right\} + 2\Delta y \left\{ \frac{\partial s}{\partial y}(x_0, y_0, z_0) \right\} \\ &+ 2\Delta z \left\{ \frac{\partial s}{\partial z}(x_0, y_0, z_0) \right\} + \\ &\cdots + \sum_{n_1, n_2, n_3} \frac{2}{n_1!n_2!n_3!} \frac{\partial^{(2k-1)} s}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} \Delta x^{n_1} \Delta y^{n_2} \Delta z^{n_3} \\ &\text{(here } 2k-1 = n_1 + n_2 + n_3) \\ &\vdots \quad \vdots \quad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{j=1}^k \left[\left\{ h^{(2k+1)}(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - h^{(2k+1)}(x_0 - \Delta x, y_0 - \Delta y, z_0 - \Delta z) \right\} \right. \\
&\quad \left. - \left\{ g^{(2k-1)}(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - g^{(2k-1)}(x_0 - \Delta x, y_0 - \Delta y, z_0 + \Delta z) \right\} \right] = \\
&\quad \sum_{n_1, n_2, n_3} \frac{2}{n_1!n_2!n_3!} \frac{\partial^3 s}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} \Delta x^{n_1} \Delta y^{n_2} \Delta z^{n_3} \\
&\quad \text{(here } 3 = n_1 + n_2 + n_3) \\
&= + \cdots + \sum_{n_1, n_2, n_3} \frac{2}{n_1!n_2!n_3!} \frac{\partial^{(2k+1)} s}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} \Delta x^{n_1} \Delta y^{n_2} \Delta z^{n_3} \\
&\quad \text{(here } 2k+1 = n_1 + n_2 + n_3)
\end{aligned}$$

We define following three force of mortality functions evaluated at (x_0, y_0, z_0) :

$$\begin{aligned}
\mu_x(x_0, y_0, z_0) &= -\frac{1}{s(x_0, y_0, z_0)} \frac{\partial s(x_0, y_0, z_0)}{\partial x} \\
\mu_y(x_0, y_0, z_0) &= -\frac{1}{s(x_0, y_0, z_0)} \frac{\partial s(x_0, y_0, z_0)}{\partial y} \\
\mu_z(x_0, y_0, z_0) &= -\frac{1}{s(x_0, y_0, z_0)} \frac{\partial s(x_0, y_0, z_0)}{\partial z}
\end{aligned}$$

and further we define three functions of forces of mortality as follows:

$$\begin{aligned}
\mu(x_0 + \Delta x, y_0, z_0) &= -\frac{1}{s(x_0 + \Delta x, y_0, z_0)} \frac{\partial s(x_0 + \Delta x, y_0, z_0)}{\partial x} \\
\mu(x_0, y_0 + \Delta y, z_0) &= -\frac{1}{s(x_0, y_0 + \Delta y, z_0)} \frac{\partial s(x_0, y_0 + \Delta y, z_0)}{\partial y} \\
\mu(x_0, y_0, z_0 + \Delta z) &= -\frac{1}{s(x_0, y_0, z_0 + \Delta z)} \frac{\partial s(x_0, y_0, z_0 + \Delta z)}{\partial z}
\end{aligned}$$

Using the above definitions, we obtain following rates evaluated at (x_0, y_0, z_0) :

$$\begin{aligned}
\frac{\partial \mu(x, y, z)}{\partial x} &= \lim_{\Delta x \rightarrow 0} \frac{\mu(x_0 + \Delta x, y_0, z_0) - \mu(x_0, y_0, z_0)}{\Delta x} \\
\frac{\partial \mu(x, y, z)}{\partial y} &= \lim_{\Delta y \rightarrow 0} \frac{\mu(x_0, y_0 + \Delta y, z_0) - \mu(x_0, y_0, z_0)}{\Delta y} \\
\frac{\partial \mu(x, y, z)}{\partial z} &= \lim_{\Delta z \rightarrow 0} \frac{\mu(x_0, y_0, z_0 + \Delta z) - \mu(x_0, y_0, z_0)}{\Delta z}
\end{aligned}$$

By taking pairs of limits $(\Delta y \rightarrow 0, \Delta z \rightarrow 0)$, $(\Delta x \rightarrow 0, \Delta z \rightarrow 0)$, and $(\Delta x \rightarrow 0, \Delta y \rightarrow 0)$, separately in the equation (4.1), we obtain following three equations:

$$\begin{aligned}
s(x_0 + \Delta x, y_0, z_0) &= s(x_0, y_0, z_0) + \Delta x \left\{ \frac{\partial s}{\partial x}(x_0, y_0, z_0) \right\} + \cdots + \frac{\Delta x^n}{n!} \left\{ \frac{\partial^n s}{\partial x^n}(x_0, y_0, z_0) \right\} + \cdots
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
s(x_0, y_0 + \Delta y, z_0) &= s(x_0, y_0, z_0) + \Delta y \left\{ \frac{\partial s}{\partial y}(x_0, y_0, z_0) \right\} + \cdots + \frac{\Delta y^n}{n!} \left\{ \frac{\partial^n s}{\partial y^n}(x_0, y_0, z_0) \right\} + \cdots
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
s(x_0, y_0, z_0 + \Delta z) &= s(x_0, y_0, z_0) + \Delta z \left\{ \frac{\partial s}{\partial z}(x_0, y_0, z_0) \right\} + \cdots + \frac{\Delta z^n}{n!} \left\{ \frac{\partial^n s}{\partial z^n}(x_0, y_0, z_0) \right\} + \cdots
\end{aligned} \tag{4.5}$$

By taking pairs of limits $(\Delta y \rightarrow 0, \Delta z \rightarrow 0)$, $(\Delta x \rightarrow 0, \Delta z \rightarrow 0)$, and $(\Delta x \rightarrow 0, \Delta y \rightarrow 0)$, separately in the equation (4.2), we obtain following three equations:

$$\begin{aligned}
s(x_0 - \Delta x, y_0, z_0) &= s(x_0, y_0, z_0) - \Delta x \left\{ \frac{\partial s}{\partial x}(x_0, y_0, z_0) \right\} + \cdots + (-1)^n \frac{\Delta x^n}{n!} \left\{ \frac{\partial^n s}{\partial x^n}(x_0, y_0, z_0) \right\} + \cdots
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
s(x_0, y_0 - \Delta y, z_0) &= s(x_0, y_0, z_0) - \Delta y \left\{ \frac{\partial s}{\partial y}(x_0, y_0, z_0) \right\} + \cdots + (-1)^n \frac{\Delta y^n}{n!} \left\{ \frac{\partial^n s}{\partial y^n}(x_0, y_0, z_0) \right\} + \cdots
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
s(x_0, y_0, z_0 - \Delta z) &= s(x_0, y_0, z_0) - \Delta z \left\{ \frac{\partial s}{\partial z}(x_0, y_0, z_0) \right\} + \cdots + (-1)^n \frac{\Delta z^n}{n!} \left\{ \frac{\partial^n s}{\partial z^n}(x_0, y_0, z_0) \right\} + \cdots
\end{aligned} \tag{4.8}$$

Suppose $h_1^n(x_0 + \Delta x, y_0, z_0)$, $h_1^n(x_0 - \Delta x, y_0, z_0)$; $h_2^n(x_0, y_0 + \Delta y, z_0)$, $h_2^n(x_0, y_0 - \Delta y, z_0)$ and $h_3^n(x_0, y_0, z_0 + \Delta z)$, $h_3^n(x_0, y_0, z_0 - \Delta z)$ for $n = 1, 2, 3, \dots$ denote the functions by ignoring the terms from the order $(n + 1)$ in the pairs of equations (??), (4.6); (4.4), (4.7) and (4.5), (4.8), then we will obtain following three series of sequences of difference functions:

$$\begin{aligned}
h_1^{(1)}(x_0 + \Delta x, y_0, z_0) - h_1^{(1)}(x_0 - \Delta x, y_0, z_0) &= 2\Delta x \left\{ \frac{\partial s}{\partial x}(x_0, y_0, z_0) \right\} \\
h_1^{(3)}(x_0 + \Delta x, y_0, z_0) - h_1^{(3)}(x_0 - \Delta x, y_0, z_0) &= 2\Delta x \left\{ \frac{\partial s}{\partial x}(x_0, y_0, z_0) \right\} + \frac{2\Delta x^3}{3!} \left\{ \frac{\partial^3 s}{\partial x^3}(x_0, y_0, z_0) \right\} \\
&\vdots \quad \vdots \quad \vdots \\
h_1^{(2k-1)}(x_0 + \Delta x, y_0, z_0) - h_1^{(2k-1)}(x_0 - \Delta x, y_0, z_0) &= 2\Delta x \left\{ \frac{\partial s}{\partial x}(x_0, y_0, z_0) \right\} + \frac{2\Delta x^3}{3!} \left\{ \frac{\partial^3 s}{\partial x^3}(x_0, y_0, z_0) \right\} + \\
&\quad \cdots + \frac{2\Delta x^{(2k-1)}}{(2k-1)!} \left\{ \frac{\partial^{(2k-1)} s}{\partial x^{(2k-1)}}(x_0, y_0, z_0) \right\} \\
&\vdots \quad \vdots \quad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^k \left[\left\{ h_1^{(2j+1)}(x_0 + \Delta x, y_0, z_0) - h_1^{(2j+1)}(x_0 - \Delta x, y_0, z_0) \right\} - \right. \\
\left. \left\{ h_1^{(2j-1)}(x_0 + \Delta x, y_0, z_0) - h_1^{(2j-1)}(x_0 - \Delta x, y_0, z_0) \right\} \right] &= \frac{2\Delta x^3}{3!} \left\{ \frac{\partial^3 s}{\partial x^3}(x_0, y_0, z_0) \right\} + \\
(4.9) \quad &\quad \cdots + \frac{2\Delta x^{(2k+1)}}{(2k+1)!} \left\{ \frac{\partial^{(2k+1)} s}{\partial x^{(2k+1)}}(x_0, y_0, z_0) \right\}
\end{aligned}$$

$$\begin{aligned}
h_2^{(1)}(x_0, y_0 + \Delta y, z_0) - h_2^{(1)}(x_0, y_0 - \Delta y, z_0) &= 2\Delta y \left\{ \frac{\partial s}{\partial y}(x_0, y_0, z_0) \right\} \\
h_2^{(3)}(x_0, y_0 + \Delta y, z_0) - h_2^{(3)}(x_0, y_0 - \Delta y, z_0) &= 2\Delta y \left\{ \frac{\partial s}{\partial y}(x_0, y_0, z_0) \right\} + \frac{2\Delta y^3}{3!} \left\{ \frac{\partial^3 s}{\partial y^3}(x_0, y_0, z_0) \right\} \\
&\vdots \quad \vdots \quad \vdots \\
h_2^{(2k-1)}(x_0, y_0 + \Delta y, z_0) - h_2^{(2k-1)}(x_0, y_0 - \Delta y, z_0) &= 2\Delta y \left\{ \frac{\partial s}{\partial y}(x_0, y_0, z_0) \right\} + \frac{2\Delta y^3}{3!} \left\{ \frac{\partial^3 s}{\partial y^3}(x_0, y_0, z_0) \right\} \\
&\quad \cdots + \frac{2\Delta y^{2k-1}}{(2k-1)!} \left\{ \frac{\partial^{2k-1} s}{\partial y^{2k-1}}(x_0, y_0, z_0) \right\} \\
&\vdots \quad \vdots \quad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^k \left[\left\{ h_2^{(2j+1)}(x_0, y_0 + \Delta y, z_0) - h_2^{(2j+1)}(x_0, y_0 - \Delta y, z_0) \right\} - \right. \\
\left. \left\{ h_2^{(2j-1)}(x_0, y_0 + \Delta y, z_0) - h_2^{(2j-1)}(x_0, y_0 - \Delta y, z_0) \right\} \right] &= \frac{2\Delta y^3}{3!} \left\{ \frac{\partial^3 s}{\partial y^3}(x_0, y_0, z_0) \right\} + \\
(4.10) \quad &\cdots + \frac{2\Delta y^{2k+1}}{(2k+1)!} \left\{ \frac{\partial^{2k+1} s}{\partial y^{2k+1}}(x_0, y_0, z_0) \right\}
\end{aligned}$$

$$\begin{aligned}
h_3^{(1)}(x_0, y_0, z_0 + \Delta z) - h_3^{(1)}(x_0, y_0, z_0 - \Delta z) &= 2\Delta z \left\{ \frac{\partial s}{\partial z}(x_0, y_0, z_0) \right\} \\
h_3^{(3)}(x_0, y_0, z_0 + \Delta z) - h_3^{(3)}(x_0, y_0, z_0 - \Delta z) &= 2\Delta z \left\{ \frac{\partial s}{\partial z}(x_0, y_0, z_0) \right\} + \frac{2\Delta z^3}{3!} \left\{ \frac{\partial^3 s}{\partial z^3}(x_0, y_0, z_0) \right\} \\
&\vdots \quad \vdots \quad \vdots \\
h_3^{(2k-1)}(x_0, y_0, z_0 + \Delta z) - h_3^{(2k-1)}(x_0, y_0, z_0 - \Delta z) &= 2\Delta z \left\{ \frac{\partial s}{\partial z}(x_0, y_0, z_0) \right\} + \frac{2\Delta z^3}{3!} \left\{ \frac{\partial^3 s}{\partial z^3}(x_0, y_0, z_0) \right\} \\
&\cdots + \frac{2\Delta z^{(2k-1)}}{(2k-1)!} \left\{ \frac{\partial^{(2k-1)} s}{\partial z^{(2k-1)}}(x_0, y_0, z_0) \right\} \\
&\vdots \quad \vdots \quad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^k \left[\left\{ h_3^{(2j+1)}(x_0, y_0, z_0 + \Delta z) - h_3^{(2j+1)}(x_0, y_0, z_0 - \Delta z) \right\} - \right. \\
\left. \left\{ h_3^{(2j-1)}(x_0, y_0, z_0 + \Delta z) - h_3^{(2j-1)}(x_0, y_0, z_0 - \Delta z) \right\} \right] &= \frac{2\Delta z^3}{3!} \left\{ \frac{\partial^3 s}{\partial z^3}(x_0, y_0, z_0) \right\} + \\
(4.11) \quad &\cdots + \frac{2\Delta z^{(2k+1)}}{(2k+1)!} \left\{ \frac{\partial^{(2k+1)} s}{\partial z^{(2k+1)}}(x_0, y_0, z_0) \right\}
\end{aligned}$$

The rate of changes in the survival function with respect to one variable and corresponding forces of mortalities for three variables can be obtained using the following derivations.

$$\frac{\partial s(x_0 + \Delta x, y_0, z_0)}{\partial x} = \frac{\partial s(x_0, y_0, z_0)}{\partial x} + \Delta x \left\{ \frac{\partial^2 s}{\partial x^2}(x_0, y_0, z_0) \right\} + \cdots + \frac{\Delta x^n}{n!} \left\{ \frac{\partial^{n+1} s}{\partial x^{n+1}}(x_0, y_0, z_0) \right\} + \cdots$$

$$\frac{\partial s(x_0, y_0 + \Delta y, z_0)}{\partial y} = \frac{\partial s(x_0, y_0, z_0)}{\partial y} + \Delta y \left\{ \frac{\partial^2 s}{\partial y^2}(x_0, y_0, z_0) \right\} + \cdots + \frac{\Delta y^n}{n!} \left\{ \frac{\partial^{n+1} s}{\partial y^{n+1}}(x_0, y_0, z_0) \right\} + \cdots$$

$$\frac{\partial s(x_0, y_0, z_0 + \Delta z)}{\partial z} = \frac{\partial s(x_0, y_0, z_0)}{\partial z} + \Delta z \left\{ \frac{\partial^2 s}{\partial z^2}(x_0, y_0, z_0) \right\} + \cdots + \frac{\Delta z^n}{n!} \left\{ \frac{\partial^{n+1} s}{\partial z^{n+1}}(x_0, y_0, z_0) \right\} + \cdots$$

$$\mu(x_0 + \Delta x, y_0, z_0) = -\frac{1}{s(x_0, y_0, z_0) + \sum_{i=1}^{\infty} \left[\frac{\Delta x^i}{i!} \left\{ \frac{\partial^i s}{\partial x^i}(x_0, y_0, z_0) \right\} \right]} \sum_{i=0}^{\infty} \left[\frac{\Delta x^i}{i!} \left\{ \frac{\partial^{i+1} s}{\partial x^{i+1}}(x_0, y_0, z_0) \right\} \right]$$

$$\mu(x_0, y_0 + \Delta y, z_0) = -\frac{1}{s(x_0, y_0, z_0) + \sum_{i=1}^{\infty} \left[\frac{\Delta y^i}{i!} \left\{ \frac{\partial^i s}{\partial y^i}(x_0, y_0, z_0) \right\} \right]} \sum_{i=0}^{\infty} \left[\frac{\Delta y^i}{i!} \left\{ \frac{\partial^{i+1} s}{\partial y^{i+1}}(x_0, y_0, z_0) \right\} \right]$$

$$\mu(x_0, y_0, z_0 + \Delta z) = -\frac{1}{s(x_0, y_0, z_0) + \sum_{i=1}^{\infty} \left[\frac{\Delta z^i}{i!} \left\{ \frac{\partial^i s}{\partial z^i}(x_0, y_0, z_0) \right\} \right]} \sum_{i=0}^{\infty} \left[\frac{\Delta z^i}{i!} \left\{ \frac{\partial^{i+1} s}{\partial z^{i+1}}(x_0, y_0, z_0) \right\} \right]$$

$$\begin{aligned} \frac{\partial \mu(x, y, z)}{\partial x} &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{1}{s(x_0, y_0, z_0)} \left\{ \frac{h_1^{(1)}(x_0 + \Delta x, y_0, z_0) - h_1^{(1)}(x_0 - \Delta x, y_0, z_0)}{2\Delta x} \right\} \right. \\ &\quad \left. - \left\{ \frac{\sum_{i=0}^{\infty} \left[\frac{\Delta x^i}{i!} \left\{ \frac{\partial^{i+1} s}{\partial x^{i+1}}(x_0, y_0, z_0) \right\} \right]}{s(x_0, y_0, z_0) + \sum_{i=1}^{\infty} \left[\frac{\Delta x^i}{i!} \left\{ \frac{\partial^i s}{\partial x^i}(x_0, y_0, z_0) \right\} \right]} \right\} \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial \mu(x, y, z)}{\partial y} &= \lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} \left[\frac{1}{s(x_0, y_0, z_0)} \left\{ \frac{h_2^{(1)}(x_0, y_0 + \Delta y, z_0) - h_2^{(1)}(x_0, y_0 - \Delta y, z_0)}{2\Delta y} \right\} \right. \\ &\quad \left. - \left\{ \frac{\sum_{i=0}^{\infty} \left[\frac{\Delta y^i}{i!} \left\{ \frac{\partial^{i+1} s}{\partial y^{i+1}}(x_0, y_0, z_0) \right\} \right]}{s(x_0, y_0, z_0) + \sum_{i=1}^{\infty} \left[\frac{\Delta y^i}{i!} \left\{ \frac{\partial^i s}{\partial y^i}(x_0, y_0, z_0) \right\} \right]} \right\} \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial \mu(x, y, z)}{\partial z} &= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left[\frac{1}{s(x_0, y_0, z_0)} \left\{ \frac{h_3^{(1)}(x_0, y_0, z_0 + \Delta z) - h_3^{(1)}(x_0, y_0, z_0 - \Delta z)}{2\Delta z} \right\} \right. \\ &\quad \left. - \left\{ \frac{\sum_{i=0}^{\infty} \left[\frac{\Delta z^i}{i!} \left\{ \frac{\partial^{i+1} s}{\partial z^{i+1}}(x_0, y_0, z_0) \right\} \right]}{s(x_0, y_0, z_0) + \sum_{i=1}^{\infty} \left[\frac{\Delta z^i}{i!} \left\{ \frac{\partial^i s}{\partial z^i}(x_0, y_0, z_0) \right\} \right]} \right\} \right] \end{aligned}$$

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